Type of dynamic phase transition in bistable equations

Gregory Berkolaiko*
Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA

Michael Grinfeld†
Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, United Kingdom
(Received 31 May 2007; published 10 December 2007)

We consider a class of bistable periodically perturbed ordinary differential equations of importance in mathematical physics and derive an asymptotic criterion for the existence of a tricritical point (TCP). Surprisingly, in the adiabatic limit the criterion is local and very simple. It also allows one to calculate the location of a TCP in parameter space, which we illustrate with three examples.

DOI: 10.1103/PhysRevE.76.061110 PACS number(s): 05.70.Jk, 64.60.Kw, 75.10.Hk

I. INTRODUCTION

Phase transitions in nonequilibrium systems have attracted much attention in the past few decades. In this paper we address the question of the type of phase transition seen in systems placed in oscillating field. While there have been many investigations of such systems using Monte Carlo simulations [1–3] and while the phenomenon has been observed experimentally [4], to achieve progress by analytical tools one has to, first of all, study mean-field models. It is in the mean-field context that dynamic phase transition (DPT) was first discussed by Tomé and de Oliveira [5] (see also Mendes and Lage [6]). Other analytical investigations of the nature of the DPT include [7–9].

In this paper we address the type of dynamic phase transition undergone, and the existence of the tricritical point in equations of the general type

$$\epsilon x' = f(x,\beta) + h \cos 2\pi t, \quad x \in \mathbb{R}, \tag{1}$$

where, for all β, f is an odd function of x with at most three zeros. If $f(x,\beta)$ has exactly three zeros, it is called *bistable* [see assumptions (A i)–(A iii) for precise conditions on f]. This type of equation arises in many different mean-field models. For example, in the context of the ferromagnetic Ising model in an oscillating magnetic field, x would correspond to the mean magnetization, β is proportional to the inverse of the temperature of the sample, h is the amplitude of the external field, and ϵ is the time relaxation parameter. In the present paper we study the small ϵ behavior which corresponds to low-frequency external field.

We shall now attempt to explain the phenomenon of dynamic phase transition using the model equation

$$\epsilon x' = -x + \tanh(\beta x) + h \cos 2\pi t. \tag{2}$$

This equation is equivalent to the Suzuki-Kubo equation, Eq. (2.9) of [10]; for the changes of variables required to put the Suzuki-Kubo equation into the form (2), the reader is referred to [9]. This equation was also studied by Tomé and de Oliveira [5]. First of all, setting h to zero (no external field) we observe two stable stationary *nonzero* solutions when $\beta > 1$. Decreasing β (which corresponds to increasing the temperature), we see that at $\beta=1$ the equilibrium solutions merge in a supercritical pitchfork bifurcation producing a single stable equilibrium solution at zero: $f(x,\beta)$ ceases to be bistable. The temperature corresponding to the value 1 of our parameter β is the Curie temperature and the transition described is the ferro-para phase transition. It is important for the forthcoming discussion that if the parameter β is *increased*, the transition from paramagnetic to ferromagnetic phase happens at the same value of β, $\beta=1$.

Now we fix $\beta > 1$ and switch on the external field, $h > 0$. By the implicit function theorem, stationary solutions now become periodic of period ones. For small values of h there are two stable periodic cycles oscillating around the stationary solutions of Eq. (2) with $h=0$. However, it can be shown (for a proof, see [9]), that for large h there is only one stable periodic solution (cycle) and that its average $\bar{x}$ over a period is zero:

$$\bar{x} = \int_0^1 x(t)dt = 0.$$

An important question is to understand how the two stable cycles (and an unstable one which is not normally seen in numerical simulations unless one performs calculations in reverse time) merge to produce a single stable cycle. Two possible minimal scenarios are depicted in Fig. 1, with h as a parameter and $\bar{x}$ as the dependent variable.

The difference between the two scenarios is of high physical relevance. Assume that we start with a stable periodic solution and the parameter h is being increased adiabatically. This solution has nonzero mean; we will call it the ferromagnetic cycle. As h is increased we will see a transition between ferromagnetic and paramagnetic (zero-mean) cycles which happens at some value of h which we will denote by h_{fp}. It has been observed by Tomé and de Oliveira [5] that for small temperature (large β) this transition happens discontinuously. Moreover, for these values of β, the transition between paramagnetic and ferromagnetic cycles when h is being *decreased* happens at value h_{pf} which is *smaller* than h_{fp}. This is consistent with the supercritical bifurcation [scenario (b) of Fig. 1] of the paramagnetic cycle, see Fig. 2 for

*Gregory.Berkolaiko@math.tamu.edu
†michael@maths.strath.ac.uk

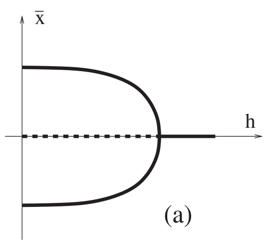

FIG. 1. Two possible minimal scenarios of the emergence of a single periodic solution. In scenario (a) three solutions merge in a subcritical pitchfork bifurcation. In scenario (b) the central solution first undergoes a supercritical pitchfork bifurcation, emitting two unstable solutions. These unstable solutions disappear in fold bifurcations upon meeting the stable solutions.

more details. The point in the β-h plane where the curves $h_{pf}(\beta)$ and $h_{fp}(\beta)$ meet is called the tricritical point (TCP). In other words, at the TCP the phase transition changes from being continuous to being discontinuous.

An alternative mean-field theory in the Ising context leading to an equation of the type

$$\epsilon x' = \lambda x - \mu x^3 + h \cos 2\pi t \tag{3}$$

(also known as the optical bistability equation [11]) was proposed by Zimmer [7], who argued that only continuous phase transition can occur. In Monte Carlo simulations, some authors (e.g., [1,12]) detected the TCP and some authors (e.g., [3]) disputed its existence. For more references and a review, see [13].

In this work, rather than address the adequacy of different mean-field theories, we will analytically derive a simple criterion on f which determines whether or not a TCP exists in an equation of the general type (1). In cases where TCP exists, our approach also allows us to calculate its position in the $\epsilon \to 0$ limit.

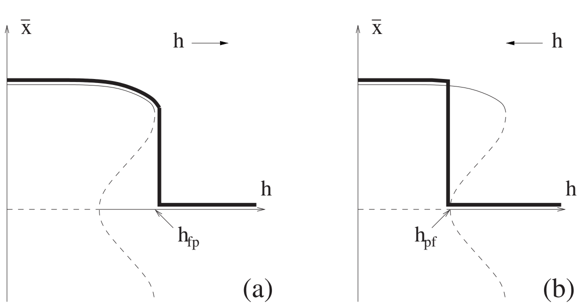

FIG. 2. If the bifurcation of the zero-mean solution is a supercritical pitchfork [scenario (b) of Fig. 1], one can effect a discontinuous change (a first-order phase transition) on the stable periodic cycle by adiabatically changing h. Here the fragment of the bifurcation diagram is drawn in thin lines and the evolution of the mean of the periodic cycle is indicated by the thick lines. When increasing the h, the transition from ferro (nonzero-mean) to para (zero-mean) cycle happens at $h = h_{fp}$ [part (a)]. However, if h is decreased the para-ferro transition happens at a different value, $h = h_{pf} \neq h_{fp}$ [part (b)].

II. STATEMENT OF THE MAIN RESULT

In [9] we have considered the Suzuki-Kubo equation (2) and the optical bistability equation (3). The main result of [9] is that Eq. (3) has at most three periodic solutions, while in Eq. (2) for sufficiently large β, there is an interval of values of h for which it has at least five periodic solutions.

However, the approach of [9] does not make any use of the additional parameter of the problem, the relaxation time ϵ. In the present paper we consider a general bistable equation,

$$\epsilon x' = f(x) + h \cos 2\pi t, \tag{4}$$

with precise assumptions on $f(x)$ to be specified later, and derive a very simple criterion on f which ensures that *for* $\epsilon > 0$ *small enough* there is an interval of values of h for which Eq. (4) has at least five periodic solutions. Amazingly, the type of such a global event as bifurcation of periodic solutions is governed by the behavior of $f(x)$ at one point only.

The precise assumptions on $f(x)$ used below are as follows:

(A i): $f(x) = -f(-x)$;

(A ii): there is a unique value $\alpha > 0$ such that $f(\alpha) = f(0) = f(-\alpha) = 0$; $f(x) < 0$ for $x > \alpha$;

(A iii): there is a unique value $a > 0$, such that $f'(a) = 0$.

Since $f(x)$ is an odd function by (A i), by (A iii) it has a global maximum in the region $x > 0$ at a. We find that if $f'''(a) > 0$ then, for small ϵ, the bifurcation of the unstable symmetric solution is a supercritical pitchfork [Fig. 1(b)] and therefore the corresponding phase transition is discontinuous. If $f'''(a) < 0$ then the bifurcation is a subcritical pitchfork [Fig. 1(a)] and the phase transition is (locally) continuous.

If the function $f(x)$ depends on a parameter β, then our criterion can be used to investigate existence of the TCP and, moreover, to find the location of the TCP in the $\epsilon \to 0$ limit. Essentially, the TCP is located at the value of β where the function $f_{xxx}(a(\beta), \beta)$ changes sign.

The structure of the remainder of the paper is as follows. After collecting the necessary definitions and results from [9], in Sec. IV we derive the above criterion for having at least five periodic solutions as $\epsilon \to 0$; this criterion is then used in Sec. V to derive the value(s) of β for which the tricritical point occurs as $\epsilon \to 0$ in the Suzuki-Kubo equation (2). In that section we also show that even an equation (4) with $f(x)$ being the simplest "correct" Padé approximant of $\tanh(\beta x) - x$ (the [3/2] one) correctly reproduces the (local) bifurcation behavior of Eq. (2).

III. PRELIMINARY RESULTS

It is proved in [[9], Theorem 2.1] that there exists a value of h, h_0, such that for all $h > h_0$ Eq. (4) has precisely one periodic solution (which from symmetry considerations is then necessarily Liapunov stable and has mean zero).

We want to understand the nature of the bifurcation that the zero-mean solution undergoes at the value $h = h_{cs}$ where it becomes stable never to lose stability again as we increase h. To achieve our aim, we apply the Liapunov-Schmidt reduction (LSR). The reduction, around a particular solution $x_0(t)$, leads to the construction of a reduced function g: $\mathbb{R}^3 \to \mathbb{R}$ such that the solutions of $g(y,h,\epsilon)=0$ are locally in one-to-one correspondence with the solutions of the original equation. Under this correspondence the solution $x_0(t)$ is mapped into the zero solution of $g(y,h,\epsilon)=0$. It is rarely possible to compute $g(y,h,\epsilon)$ explicitly, but one can examine the bifurcation picture around x_0 by computing the derivatives of g. For more details on the reduction we refer the reader to [14].

In [9] it is shown that, due to the symmetry of $f(x)$, the bifurcation of the zero-mean solution has to be a pitchfork. Thus at the bifurcation point we have the criticality condition $g_y=0$, and $g_{yy}=0$.

Furthermore, since it is the stability-gaining bifurcation, $g_{yh}\geq 0$ and hence the direction of bifurcation in the nondegenerate case is determined by the sign of g_{yyy}. The relevant formulas derived by the LSR in [9] are

$$g_y = \int_0^1 f'(x_0(t))dt \equiv 0; \tag{5}$$

$$g_{yyy} = -\int_0^1 f'''(x_0(t))\exp\left(\frac{2}{\epsilon}\int_0^t f'(x_0(s))ds\right)dt, \tag{6}$$

where $x_0(t)$ is the zero-mean solution undergoing bifurcation. Here Eq. (5) should be considered as a condition on $x_0(t)$ to be undergoing a bifurcation (a "criticality condition"). If g_{yyy} is negative, the bifurcation at $h=h_{cs}$ is a supercritical pitchfork, the scenario shown on Fig. 1(b). Hence the main question we need to ask ourselves is, under what condition on $f(x)$ is g_{yyy} negative? It turns out that, under the assumption that ϵ is sufficiently small, the answer to this question is very simple.

IV. DIRECTION OF THE PITCHFORK AS $\epsilon \to 0$

First of all, we need to characterize the critical solution $x_0(t)$. Note that $x_0(t+1/2)=-x(t)$. As before, we let a be the point of global maximum of $f(x)$ for $x>0$.

We observe that there are values $t_\pm^j \in (0,1)$, $j=1,2$, such that $x_0(t_\pm^j)=\pm a$. This is obvious as otherwise the criticality condition (5) cannot be met.

Now let us set

$$\phi(t) = \int_0^t f'(x_0(s))ds. \tag{7}$$

Then at $t_\pm^j$ we have $\phi'(t)=f'(x_0(t))=0$ by definition of $t_\pm^j$ and a. Furthermore,

$$\phi''(t) = f''(x_0(t))x_0'(t).$$

Hence $\phi(t)$ reaches a maximum twice over a period of the solution, at points t^* and (by symmetry) $t^*+1/2$. Since t^* is one of the points of the set $\{t_\pm^j\}$, we have $x_0(t^*)=a$. Also, $\phi(t^*)=\phi(t^*+1/2)$ and $\phi''(t^*)=\phi''(t^*+1/2)$.

Now we apply the method of Laplace [15], as $\epsilon\to 0$, to the integral in Eq. (6). We compute the value of g_{yyy} as

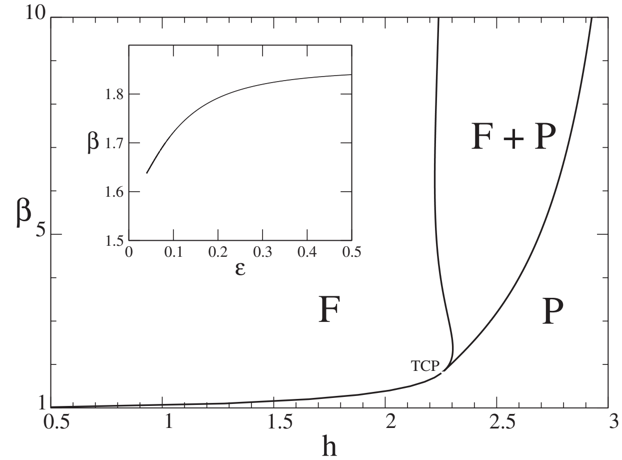

FIG. 3. Dependence of the type of the periodic solutions on the parameters β and h in Eq. (2) with $\epsilon=1/2$. F marks the region of existence of a nonzero-mean (ferromagnetic) stable periodic solution and P marks the region with a stable zero-mean solution (paramagnetic). For large values of β there is a range of h in which F and P solutions can co-exist. The point at which this range shrinks to zero is the tricritical point (TCP). Inset: location of the TCP as a function of ϵ. For small values of ϵ the computation becomes unstable [5] due to the exponential narrowing of the overlap (F+P) region [9].

$$g_{yyy} = -2\sqrt{2\pi}\frac{\exp[(2/\epsilon)\phi(t^*)]f'''(a)}{\sqrt{-(2/\epsilon)\phi''(t^*)}}[1+O(\epsilon)]. \tag{8}$$

From Eq. (8) we see that as $\epsilon\to 0$, amazingly, the direction of the pitchfork bifurcation is determined solely by the sign of $f'''(a)$: if this is positive, the bifurcation is supercritical and we will have an interval of values of h where Eq. (4) has five periodic solutions. We formulate this as a theorem.

Theorem 4.2 As $\epsilon\to 0$, under the assumptions (A i)–(A iii), the stability-gaining bifurcation of the symmetric solution at $h=h_{cs}$ is a subcritical bifurcation if $f'''(a)<0$ and it is a supercritical pitchfork if $f'''(a)>0$.

V. EXAMPLES

Example 1. If $f(x)=\lambda x-\mu x^3$, see Eq. (3), with positive λ and μ then $f'''(a)=-6\mu$ is always negative, again confirming that the bifurcation of the symmetric solution must be a subcritical pitchfork.

Example 2. Let us consider again the Suzuki-Kubo equation (2) and compute the location, as $\epsilon\to 0$, of the tricritical point (see Fig. 3), that is, the value of $\beta>1$ such that the bifurcation flips from being sub- to supercritical.

In other words, we want to find the value of β, such that if $a(\beta)>0$ solves the equation $f'(a(\beta))=0$, where $f(x)=\tanh(\beta x)-x$, we also have $f'''(a(\beta))=0$. We obtain

$$a(\beta) = \frac{1}{\beta}\tanh^{-1}\left(\sqrt{\frac{\beta-1}{\beta}}\right).$$

Hence we find that $f'''(a(\beta))=0$ is equivalent to the amazingly simple expression

$$2\beta(-3+2\beta)=0,$$

so that

$$\beta_{tcp}=\frac{3}{2}+O(\epsilon).$$

Incidentally, in [9] (see Theorem 4.1 and Remark 4.2) it was shown that for $\beta<3/2$ the bifurcation is subcritical for all values of ϵ, so we expect the correction term to be positive. This prediction is verified numerically in the inset of Fig. 3.

Example 3. Let us consider the simplest "correct" Padé approximant of the right-hand side of the Suzuki-Kubo equation (2). To be correct we would like it to have the same (linear) rate of growth at infinity as the Suzuki-Kubo equation itself. Thus the simplest such approximant would be the [3/2] Padé,

$$f(x)=\frac{1}{3}\frac{x[15(\beta-1)+x^2(\beta^3-6\beta^2)]}{5+2\beta^2x^2}.$$

Following the same procedure as before, we find that Eq. (4) with the above right-hand side also has a tricritical point and that

$$\beta_{tcp}=1.493\,366\,856+O(\epsilon).$$

We remind the reader that the cubic approximation to the Suzuki-Kubo $f(x)$ has no tricritical point, the bifurcation being always subcritical. For comparison, the fifth order McLaurin series truncation of the Suzuki-Kubo $f(x)$ is not in general bistable at all.

VI. CONCLUSIONS

We have developed a simple criterion to detect if an equation of the type (1) has a discontinuous phase transition for a physically important class of nonlinearities $f(x)$. Applying this criterion allows one to check for existence of a tricritical point and even find its location.

It would be interesting to extend our theory to cover the mean-field theory derived for the Blume-Capel model by Keskin *et al.* [16]. The results of [16] seem to predict a more complicated bifurcation diagram than the minimal ones of Fig. 1, with multiple tricritical points. Another obvious extension would be to relax the somewhat restrictive assumption (A iii). In particular, if the range of the bifurcating solution $x_0(t)$ is $[-A,A]$, the behavior of $f(x)$ for $|x|>A$ is not relevant to the bifurcation picture. Furthermore, we left unproved the question of uniqueness of bifurcation from the zero-mean solution; it seems plausible that this is related to assumption (A iii).

ACKNOWLEDGMENTS

We would like to thank J. Carr and J. Vanneste for fruitful discussions.

[1] M. Acharyya, Phys. Rev. E **59**, 218 (1999).
[2] S. W. Sides, P. A. Rikvold, and M. A. Novotny, Phys. Rev. Lett. **81**, 834 (1998); S. W. Sides, P. A. Rikvold, and M. A. Novotny, Phys. Rev. E **59**, 2710 (1999); B. K. Chakrabarti and M. Acharyya, Rev. Mod. Phys. **71**, 847 (1999); G. Korniss, C. J. White, P. A. Rikvold, and M. A. Novotny, Phys. Rev. E **63**, 016120 (2000); A. Chatterjee and B. K. Chakrabarti, *ibid.* **67**, 046113 (2003).
[3] G. Korniss, P. A. Rikvold, and M. A. Novotny, Phys. Rev. E **66**, 056127 (2002).
[4] Q. Jiang, H.-N. Yang, and G.-C. Wang, Phys. Rev. B **52**, 14911 (1995); Q. Jiang, H.-N. Yang, and G.-C. Wang, J. Appl. Phys. **79**, 5122 (1996); W. Kleemann, T. Braun, J. Dec, and O. Petracic, Phase Transitions **78**, 811 (2005).
[5] T. Tomé and M. J. de Oliveira, Phys. Rev. A **41**, 4251 (1990).
[6] J. F. F. Mendes and E. J. S. Lage, J. Stat. Phys. **64**, 653 (1991).
[7] M. F. Zimmer, Phys. Rev. E **47**, 3950 (1993).
[8] H. Tutu and N. Fujiwara, J. Phys. Soc. Jpn. **73**, 2680 (2004).
[9] G. Berkolaiko and M. Grinfeld, Proc. R. Soc. London, Ser. A **462**, 2067 (2006).
[10] M. Suzuki and R. Kubo, J. Phys. Soc. Jpn. **24**, 51 (1968).
[11] P. Jung, G. Gray, R. Roy, and P. Mandel, Phys. Rev. Lett. **65**, 1873 (1990).
[12] M. Acharyya and B. K. Chakrabarti, Physica A **192**, 471 (1993).
[13] M. Acharyya, Int. J. Mod. Phys. C **16**, 1631 (2005).
[14] M. Golubitsky and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory* (Springer-Verlag, New York, 1985), Vol. I.
[15] E. T. Copson, *Asymptotic Expansions* (Cambridge University Press, New York, 1965).
[16] M. Keskin, O. Canko, and U. Temizer, Phys. Rev. E **72**, 036125 (2005); M. Keskin, O. Canko, and B. Deviren, *ibid.* **74**, 011110 (2006).